

Homework 10

Math 121A, Fall 2025.

Due: Nov 5, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

We will use the notation $\hat{f}(k)$ to denote the k^{th} (exponential) Fourier coefficient of f , i.e.

$$f(x) = \sum_{k=0}^{\infty} \hat{f}(k) e^{ikx}$$

Problem 1. Determine which of the following functions on the real line are periodic. If they are, determine the fundamental period; if not, explain why.

$$\sin(\pi x), e^x, e^{ix} + \sin(x), \sin(3x) + \cos(5x), \sin(3x) + \sin(\sqrt{5}x).$$

Problem 2. If $f : \mathbb{R} \rightarrow \mathbb{C}$ is periodic with period $2L$ (i.e. $f(x) = f(x + 2L)$), what are the periods of $s(x) = f(cx)$, $h(x) = f(x - t)$, and $g(x) = f(-x)$? Calculate $\hat{h}(k)$ and $\hat{g}(k)$ in terms of $\hat{f}(k)$.

Problem 3. Define a pair of inner products

$$\begin{aligned} \langle f, g \rangle &= \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx \\ \langle f, g \rangle_1 &= \int_{-1}^1 \overline{f(x)} g(x) dx. \end{aligned}$$

Compute the following inner products for the following 2π -periodic functions for all integer values of k, n . What does this say about orthogonality?

$$\langle \sin kx, \sin mx \rangle, \quad \langle \sin kx, \sin mx \rangle_1, \quad \langle e^{ikx}, e^{imx} \rangle, \quad \langle e^{ikx}, e^{imx} \rangle_1.$$

Problem 4. (a) Evaluate the (exponential) Fourier coefficients $\hat{f}(k)$ of the sawtooth wave:

$$f(x) = x, \quad -\pi \leq x < \pi.$$

Sawtooth waves are common waveforms for distorted synth and guitar sounds. Here's a cool audio/video clip of what happens when we add more harmonics (i.e. waves of higher Fourier coefficients k) to a truncated Fourier series: [Youtube link](#).

(b) Use Parseval's theorem to conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

(c) Use a similar method applied to the “sharkfin” function

$$f(x) = x^2, \quad -\pi \leq x < \pi.$$

to find the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^4}.$$

Problem 5. Show that if f has only finitely many nonzero Fourier coefficients $\hat{f}(k) \neq 0$, then f must be infinitely differentiable. Conclude that any discontinuous function on the circle (e.g. the sawtooth function) must have infinitely many nonzero Fourier coefficients.

Problem 6. Let $f : [-\pi, \pi] \rightarrow \mathbb{C}$ and consider the trigonometric Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

- (a) Show that if f is even, meaning $f(x) = f(-x)$, we have $b_k = 0$ for all k .
- (b) Show that if f is odd, meaning $f(x) = -f(-x)$ we have $a_k = 0$ for all k .
- (c) Show that every function f may be written as an average of an even function and an odd function.
- (d) Decide if the following functions are even, odd, or neither and compute their (trigonometric) Fourier series:

$$\text{i. } f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 0 & x = 0 \text{ or } \pm \pi \\ 1, & 0 < x < \pi \end{cases}$$

$$\text{ii. } f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$\text{iii. } f(x) = |x|.$$

$$\text{iv. } f(x) = \cosh x.$$

Problem 7. (*Minimizing MSE*) Suppose that the partial sums of the Fourier series of $f \in L^2[-\pi, \pi]$ are

$$S_N(x) := \sum_{k=-N}^N \hat{f}(k) e^{ikx}.$$

Show that for any coefficients d_k ,

$$\|f - S_N\|^2 \leq \left\| f - \sum_{k=-N}^N d_k e^{ink} \right\|^2,$$

i.e. the partial sums of the Fourier series minimize the mean squared error among all linear combinations of $e^{-iNx}, \dots, e^{iNx}$.¹

¹In fact, this is not unique to Fourier series. A similar statement can be made for any orthogonal basis of $L^2[-\pi, \pi]$.